ISRAEL JOURNAL OF MATHEMATICS, Vol. 62, No. 3, 1988

LIGHTLY MIXING IS CLOSED UNDER COUNTABLE PRODUCTS

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ABSTRACT

Given any countable family $T_1, T_2, \ldots$ of lightly mixing transformations, we answer a question of Friedman by showing that their direct product $T_1 \times T_2 \times \cdots$ is lightly mixing. Also, if a transformation has a generating tower of lightly mixing factors then it itself is lightly mixing.

Blum, Christianson and Quiring, in their paper *Sequence mixing and α-mixing* [BCQ], define the following property of a measure preserving map of a Lebesgue probability space (X, μ). A map T is *sequence mixing*[††] if for any set A of positive measure and for any infinite collection K of integers, the set $\bigcup_{k \in K} T^k A$ is the entire space X (measure theoretically). They then observed that sequence mixing is equivalent to T being "lightly mixing" (defined below), a term coined later by Nat Friedman to emphasize its place in the hierarchy of mixing-like properties.

Define a function $\gamma(\cdot,\cdot)$ mapping a pair of non-null sets into a number in $[0, \infty)$ by

$$\gamma(A, B) = \left[\liminf_{n \to \infty} \mu(A \cap T^{-n} B)\right] \Big/ \mu(A)\mu(B).$$

T is said to be *lightly mixing* [FT] if $\gamma(\cdot,\cdot)$ is never-zero. It is *partially mixing* [FO] if $\gamma(\cdot,\cdot) \geqq \alpha$ for some positive constant α. The largest such constant is written $\alpha(T)$.

[†] Partially supported by NSF grant DMS8501519.

[††] Nowadays called *sweeping out*.

Received October 27, 1987

Cartesian products. To show $\alpha(T)$ greater than a given number α, it suffices to obtain $\gamma(A, B) \geqq \alpha$ for A and B ranging over some dense collection of sets. Consequently if $\{T_n\}$ is a countable list of transformations then

$$\alpha(T_1 \times T_2 \times \cdots) \geqq \alpha(T_1) \cdot \alpha(T_2) \cdot \cdots$$

since the inequality is immediate on any pair of sets A and B, in the product space, which are finite dimensional rectangles. Thus the class of partially mixing transformations is closed under finite direct product. The class is not closed under countable direct product since if $T_1 = T_2 = \cdots$ then the above inequality is evidently an equality. Thus for any partially mixing T which is non-mixing, its countably infinite cartesian power is not partially mixing.

The corresponding closure question for the class of lightly mixing transformations was raised by Friedman a few years ago. It requires more care, principally due to the greater delicacy of approximation arguments. Unlike the case for partial mixing, it is not immediately clear that establishing $\gamma(A, B) > 0$ for all A and B in a dense collection of sets would imply that T is lightly mixing.

In this note we hope to show that

$$\text{LightlyMixing} \times \text{LightlyMixing} \times \cdots \subset \text{LightlyMixing}$$

by means of a conditional expectation argument. Use $(T: X, \mu)$ to abbreviate $T: (X, \mu) \to (X, \mu)$. Let A^c denote the complement of set A and let $B \triangle A$ denote symmetric difference.

A simplified proof

The main result in [BCQ] is that a countably infinite direct product of partial mixing transformations is always at least lightly mixing. Indeed, such products provide the only known examples of transformations which are lightly but not partially mixing. For our purposes we need the following result, which is both a bit stronger and has a simpler proof.

(1) If each finite product $T_1 \times \cdots \times T_n$ is lightly mixing, then the infinite product $T_1 \times T_2 \times T_3 \cdots$ is also lightly mixing.

This is consequence of the following fact.

THEOREM 2. *If $\mathscr{F}_1 \subset \mathscr{F}_2 \subset \mathscr{F}_3 \subset \cdots$ is an increasing tower of factor algebras of T whose join is the entire sigma-algebra, then*

$$T\,|_{\mathscr{F}_n} \textit{ lightly mixing for all } n \Rightarrow T \textit{ is lightly mixing}.$$

Letting $\mathscr{E}$ denote conditional expectation, this theorem is implied by the lemma below.

LEMMA 3. *Suppose that for each pair of sets A_1 and A_2 of positive measure, T has a lightly mixing factor $T|_{\mathscr{F}}$ such that*

$$\mu\{x : \mathscr{E}(A_i \mid \mathscr{F})(x) > \tfrac{1}{2}\} > 0$$

for $i \in \{1, 2\}$. Then T is lightly mixing.

PROOF. For $x \in X$ let $\mu_x(\cdot)$ denote the probability measure $\mathscr{E}(\cdot \mid \mathscr{F})(x)$. By hypothesis there exists a positive δ such that each $\mu(B_i) > 0$, where

$$B_i = \left\{x : \mu_x(A_i) > \frac{1+\delta}{2}\right\}.$$

So for any n, one has for $x \in T^{-n}(B_2)$ that $\mu_x((T^{-n}A_2)^c) < (1-\delta)/2$. Since $B_i \in \mathscr{F}$ there exists $\varepsilon > 0$ such that when n is large, $\mu(B_1 \cap T^{-n}B_2) > \varepsilon$. So for $x \in B_1 \cap T^{-n}B_2$

$$\mu_x(A_1 \cap T^{-n}A_2) \geqq 1 - \mu_x(A_1^c) - \mu_x((T^{-n}A_2)^c) > \delta.$$

Consequently

$$\mu(A_1 \cap T^{-n}A_2) \geqq \int_{B_1 \cap T^{-n}B_2} \mu_x(A_1 \cap T^{-n}A_2)\,d\mu(x) \geqq \mu(B_1 \cap T^{-n}B_2)\cdot\delta \geqq \varepsilon\delta.$$

This latter quantity does not depend on n. □

LightlyMixing $\times$ LightlyMixing $\subset$ LightlyMixing

In light of (1), to show that the class of lightly mixing transformations is closed under countable product it will do to show it closed under finite product; hence, under twofold product.

THEOREM 4. *If $(T : X, m)$ and $(\tau : \Omega, \mu)$ are lightly mixing, then their product $T \times \tau$ is lightly mixing.*

As a preliminary, we recall a fact about measure spaces. Given a subset **A** of $X \times \Omega$, for each point $\omega \in \Omega$ let A_ω be the "cross-section of **A** above ω"; the subset of X consisting of those x such that $\langle x, \omega \rangle \in$ **A**. Then for μ-a.e. $\eta \in \Omega$:

$$\text{For each positive } \delta, \qquad \mu\{\omega \in \Omega : m(A_\eta \triangle A_\omega) < \delta\} > 0.$$

This follows from standard approximation arguments, by partitioning $X \times \Omega$ into many small rectangles.

PROOF OF (4). Any set $\mathrm{A} \subset X \times \Omega$ of positive measure has some cross-section, call it B, of positive measure for which

(5) For all δ: $\mu(\Lambda[\delta]) > 0$, where $\Lambda[\delta]$ denotes the set $\{\omega : m(B \triangle A_\omega) < \delta\}$.

Given a second set $\hat{\mathrm{A}} \subset X \times \Omega$ with $m \times \mu(\hat{\mathrm{A}}) > 0$, construct the analogous set $\hat{B} \subset X$ and map $\delta \mapsto \hat{\Lambda}[\delta]$.

By the lightly mixing of T we can choose constants δ and N, sufficiently small and large respectively, for which

$$\forall n \geqq N: \qquad m(B \cap T^{-n}\hat{B}) > 3\delta.$$

For this value δ, let Λ and $\hat{\Lambda}$ denote $\Lambda[\delta]$ and $\hat{\Lambda}[\delta]$. By the lightly mixing of τ, there is a sufficiently large N (the inequality above persists if we increase N) and sufficiently small ε such that

$$\forall n \geqq N: \qquad \mu(\Lambda \cap \tau^{-n}\hat{\Lambda}) > \varepsilon.$$

The following now holds for any n greater than N.

$$\begin{aligned} m \times \mu(\mathrm{A} \cap [T \times \tau]^{-n}\hat{\mathrm{A}}) &= \int_\Omega m(A_\omega \cap T^{-n}(\hat{A}_{\tau^n\omega}))d\mu(\omega) \\ &\geqq \int_\Lambda [m(B \cap T^{-n}(\hat{A}_{\tau^n\omega})) - \delta]d\mu(\omega) \end{aligned}$$

by first shrinking Ω to Λ and then applying (5). Since $\tau^n\omega \in \hat{\Lambda}$ when $\omega \in \tau^{-n}\hat{\Lambda}$, we can integrate over a still smaller set and apply (5) to the "hatted" symbols to continue our inequality

$$\begin{aligned} &\geqq \int_{\Lambda \cap \tau^{-n}\hat{\Lambda}} [m(B \cap T^{-n}\hat{B}) - 2\delta]d\mu(\omega) \\ &\geqq \mu(\Lambda \cap \tau^{-n}\hat{\Lambda}) \cdot [3\delta - 2\delta] \geqq \varepsilon\delta. \end{aligned}$$

This last quantity does not depend on n. Since A and $\hat{\mathrm{A}}$ were arbitrary subsets of the product space, this shows $T \times \tau$ to be lightly mixing. □

K-fold lightly mixing

Generalizing the proofs to higher order is routine. For a K-vector $\mathbf{n} = \langle n_1, \ldots, n_K \rangle$ of integers, let $\| \mathbf{n} \|$ denote the minimum value of $| n_i - n_j |$

over all $i \neq j$. Say that $(T: X, m)$ is *K-fold lightly mixing* if for any sets $A_1, \ldots, A_K$ each of positive measure,

$$\liminf_{\|\mathbf{n}\| \to \infty} m(T^{-n_1}A_1 \cap \cdots \cap T^{-n_K}A_K)$$

is positive.

Stated for K sets $A_1, \ldots, A_K$ and inserting "K-fold" in front of "lightly mixing" wherever it occurs, the proof of Lemma 3 perseveres once the "$\frac{1}{2}$" in its statement is replaced by $(K-1)/K$. The lemma now implies Theorem 2 and consequently (1). Furthermore, Theorem 4 remains true; replace "3δ" by $(K+1)\delta$ and then apply (5) to the integral K times so as to conclude

$$\begin{aligned} m \times \mu([T \times \tau]^{-n_1}A_1 \cap \cdots \cap [T \times \tau]^{-n_K}A_K) \\ \geqq \int_{\tau^{-n_1}\Lambda_1 \cap \cdots \cap \tau^{-n_K}\Lambda_K} [m(T^{-n_1}B_1 \cap \cdots \cap T^{-n_K}B_K) - K\delta] d\mu(\omega) \\ \geqq \mu(\tau^{-n_1}\Lambda_1 \cap \cdots \cap \tau^{-n_K}\Lambda_K) \cdot [(K+1)\delta - K\delta] \\ \geqq \varepsilon\delta \end{aligned}$$

for all $\mathbf{n}$ with $\|\mathbf{n}\|$ sufficiently large. Thus $T \times \tau$ is K-fold lightly mixing. □

"Rényi" lightly mixing. The properties of mixing, weak mixing, and ergodicity can be formulated in terms of the sequence of numbers $\{\mu(A \cap T^{-n}B)\}_{n=1}^{\infty}$ converging to $\mu(A)\mu(B)$ in the corresponding appropriate sense. With his well-known elegant Hilbert space argument, Rényi showed that it was enough to consider $\{\mu(A \cap T^{-n}A)\}$: Convergence for all pairs $\langle A, B \rangle$ of sets was implied by convergence for all pairs $\langle A, A \rangle$. Sometimes, given a mystery transformation, it is easier to establish the Rényi version of a desired property. Our proof above would be notationally simpler if we could assume that $A_1 = \cdots = A_K$. Say that an ergodic map T is K-fold "Rényi" lightly mixing if

$$\liminf_{\|\mathbf{n}\| \to \infty} m(T^{-n_1}A_1 \cap \cdots \cap T^{-n_K}A_K) > 0$$

whenever the K sets are the same.

"Rényi" lightly mixing implies lightly mixing. Given K arbitrary sets, to establish the above inequality we may without loss of generality

(a) replace any A_k with an iterate of itself;

(b) shrink any A_k to a subset of itself;

and then establish the inequality. By ergodicity, some iterate of A_2 intersects (with positive measure) A_1. First, replace A_2 by this iterate. Then replace both A_1 and A_2 by their common intersection. Some iterate of A_3 hits $A_1 = A_2$; by this common intersection replace A_1, A_2, and A_3 in the inequality above. Continuing in this manner, eventually all the $\{A_k\}$ are one-and-the-same set of positive measure. □

QUESTION. Does the distinction between "partially mixing" and "lightly mixing" exist in the class of rank-1 maps? Can even the finite rank maps distinguish these properties? The only known lightly but not partially mixing transformations come from countably infinite direct products. Such products are unlikely to have finite rank and certainly cannot be rank-1. By way of contrast, using Ornstein's technique one can construct a rank-1 transformation which is partially mixing but not mixing.

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